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Numerical Solution for the Steady Motion of a Viscous Fluid inside a Circular Boundary Using Integral Conditions

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The problem of determining the two-dimensional steady motion of a viscous incompressible fluid which is injected radially over one small arc of a circle and ejected radially over another arc is considered and examples are given of both symmetrical and asymmetrical flows. The motion is governed by the Navier–Stokes equations and the method of solution is based on the use of truncated Fourier series representations for the stream function and vorticity in the angular polar coordinate. The Navier–Stokes equations are reduced to ordinary differential equations in the radial variable and these sets of equations are solved using finite-difference methods, but with the boundary vorticity calculated using global integral conditions rather than local finite-difference approximations. One of the objects of the investigation is to relate this method to a previous study which did not use integral conditions and also to a recent study which uses an integro-differential method which is different in concept but which also uses integral conditions. A brief review of previous work on the problem is given. Comparisons of present and previous results are excellent. © 1993 Academic Press, Inc.

1. INTRODUCTION

A recent paper [1] has described a method for determining numerical solutions of the Navier–Stokes equations using integral representations and applied it to the calculation of several internal flow problems. The integral representations come from the classical theorems of viscous fluid motion expressed in suitable forms. Thus the velocity field is obtained from the vorticity field by means of an integral which embodies the Biot–Savart law, to which must be added boundary integrals which give the contributions from the boundary conditions for the velocity. A similar but somewhat more complicated formula can be obtained for the vorticity in terms of an integral over the field, together with boundary integrals which involve the boundary vorticity and the total pressure head. The solution is therefore implicit in the sense that the wanted functions, namely the vorticity and velocity components which appear on the left-hand sides of the equations, are contained in the integrals on the right-hand sides, together with their boundary conditions. An iterative procedure is therefore necessary to obtain the solution; this is a general feature of any method of solving the nonlinear Navier–Stokes equations, however, and the method proposed is to some extent a numerically implemented version of the classical analysis in which the solution of the Navier–Stokes equations is reduced to integral representations.

One of the features of the method is that boundary values of the vorticity which are generally unknown on solid boundaries can be deduced by evaluating integrals involving the vorticity and velocity vectors throughout the computational field. In other words, the boundary vorticity is determined by applying a constraint condition derived from the integral representation, which may be called an integral condition. The general nature of this condition was pointed out by Quartapelle [2], who proposed it as a generalized form of some particular cases previously treated by Dennis *et al.* [3–7], who used separation of variables techniques to express the constraint conditions in a one-dimensional form. Dennis *et al.* used a series expansion method (sometimes called the method of series truncation, cf. Van Dyke [8, 9]) to reduce the governing partial differential equations to ordinary differential equations in the case of steady flow, or time-dependent partial differential equations for unsteady flow. A brief review of some of the contributions made by means of the series truncation method, many of which use integral conditions, has been given by Anwar and Dennis [10] and a recent paper by Dennis and Quartapelle [11] has discussed numerous applications of integral conditions.

From time to time there has, of course, been interest in the numerical use of integral representations by several workers in solving Navier–Stokes problems. Payne [12] used the Biot–Savart law to determine the velocity components for time-dependent flow past an impulsively started circular cylinder but solved the vorticity transport equation using finite-differences and calculated the boundary vorticity on the cylinder locally from the transverse velocity component near the cylinder. Mills [13] used integral formulations for the velocity components and the vorticity

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based on an application of a Green's function method. As examples he solved the same internal flow problems as Wang and Wu [1], i.e., flow inside a circle with a step boundary tangential velocity component and flow into and out of a circle with prescribed inlet–outlet velocities. Mills considered a somewhat more general case in which the flow was allowed to enter and exit at arbitrary locations, whereas Wang and Wu restricted themselves to the case of symmetrical flow about a diameter. The methods used by Mills for evaluating the integrals are quite different from those of Wang and Wu, who used fundamental solution expansions appropriate to the circular region, i.e., Fourier–series expansions in terms of the polar coordinates centred at the centre of the circle.

The present paper is devoted to a study of the inflow–outflow problem within the circular boundary. As noted by Wang and Wu, much less work has been done on flow internal to a circular boundary than on flow external to one. The problem of flow inside a circle with a step boundary tangential velocity was first considered by Kuwahara and Imai [14]. The first attempt at the inflow–outflow probem was by Rayleigh [15] who considered only slow motion in which a symmetrical flow was formed by injection of fluid into the circle along a radius over an infinitesimally small arc, with a corresponding outflow at the other extremity of the diameter. The basis of his analysis was Stokes flow (Reynolds number = 0) in which the convective terms are neglected so that the Navier–Stokes equations reduce to the biharmonic equation; for this problem the integral representations give the exact solution without any iterative procedures. The case in which the convective terms are not zero was investigated by Dennis [16]. The fluid was injected symmetrically over an arc of length 2α (Fig. 1) and flows out over a symmetrically placed arc. Solutions were obtained in the cases $\alpha = \pi/180$ (1°) and $\alpha = \pi/30$ (6°) for several Reynolds numbers $R = \alpha Ua/\nu$. Wang and Wu have also considered the case $\alpha = 6°$ by their method although it is not clear exactly what their Reynolds number Re represents. We shall return to this point later when comparisons are made.

The problem of symmetrical flow in Fig. 1 was analyzed by Mills in a more general asymmetrical form in which the inflow and outflow can take place over several arcs of arbitrary lengths subject only to the condition of conservation of mass. The arcs can be of arbitrary orientation. A typical case is shown in Fig. 2. In the present paper we have extended the series truncation method used by Dennis [16] to cover this case by generalizing the half-range Fourier sine series expansion used in the case of the problem of Fig. 1 to the full-range series of sines and cosines needed for a case such as Fig. 2. We also use integral constraints to calculate the surface vorticity, whereas Dennis [16] used the usual local finite-difference approximations. The method is therefore in between that of Dennis and the method of Wang and Wu, who also use the same integral constraints, but utilize a quite different integro-differential method of determining the vorticity and the velocity components. Wang and Wu also make use of Chebyshev integration methods for evaluating the necessary integrals when the Reynolds number is high. We have found it appropriate to use a specialized integration technique for evaluating the integral constraints in the present method. It is also interesting to note that the integral constraints are virtually the same as those applicable to solving the Navier–Stokes equations for flow through a loosely coiled tube (Dennis and Ng [17]).

In the following sections we describe the series formulation and its numerical solution subject to the integral constraints. The results of calculations are presented, mainly for the symmetrical case where comparison can be made with previous calculations of Dennis and of Wang and Wu. Comparisons with both sets of results are good (assuming our interpretation of Wang and Wu's Reynolds number Re is correct) and they indicate that the present method forms an effective bridge between the two techniques. The method is

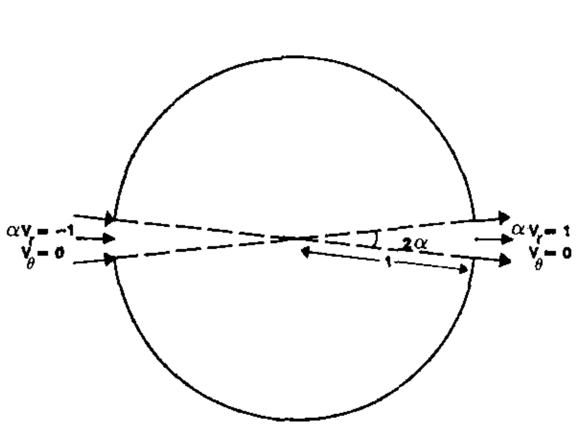

FIG. 1. Symmetrical inflow–outflow problem.

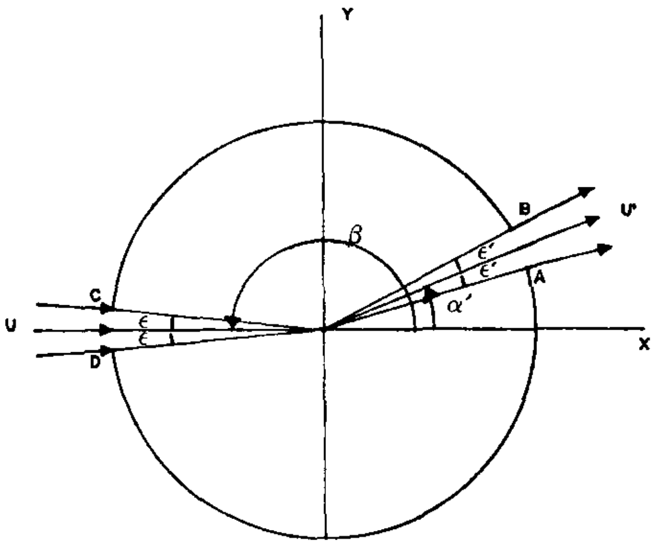

FIG. 2. Typical asymmetrical inflow–outflow problem.

quite efficient and accurate and the specialized technique of evaluating the integral constraints works well. One calculation of asymmetrical flow for a case considered by Mills is carried out and the comparison is satisfactory.

2. BASIC EQUATIONS

We shall consider the formulation for the general case of asymmetrical flow, although two specific cases are considered numerically, namely the case of symmetrical flow shown in Fig. 1 and a typical asymmetrical flow shown in Fig. 2. All quantities are assumed to be dimensionless. We follow the method of Mills and use the radius a of the circle and a typical velocity U with which to make the variables dimensionless, where $Ua\varepsilon$ is half the flow across a typical arc CD (Fig. 2). The dimensionless radial and transverse velocity components (v_r, v_θ) are given in terms of the dimensionless stream function ψ by

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta = -\frac{\partial\psi}{\partial r} \tag{1}$$

and the other dependent variable is the dimensionless scalar vorticity ζ defined by

$$\zeta = \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}. \tag{2}$$

From Eqs. (1) and (2) and from the Navier–Stokes momentum equations, the two governing equations for ψ and ζ are well known to be

$$\nabla^2\psi = \zeta, \tag{3}$$

$$\nabla^2\zeta = \frac{R}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\zeta}{\partial\theta}\right), \tag{4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

The Reynolds number in (4) is defined by

$$R = Ua\varepsilon/\nu. \tag{5}$$

This is not quite the same, apparently, as the one used by Wang and Wu. We have used a dimensionless scaling for the inlet or outlet normal velocity component so that the dimensionless flow across a given arc is independent of its length. This preserves continuity of flow in a case such as Fig. 2. The case of Fig. 2 can be identified with Fig. 1 by taking $\varepsilon = \varepsilon' = \alpha$, $\alpha' = 0$, $\beta = \pi$ (Fig. 2 actually shows $\beta = \pi$), $U = U'$. The flow is especially simple in that case, being symmetrical about the axis $y = 0$.

In the first analysis of the problem to be given, Rayleigh considered the case of Fig. 1 corresponding to $R = 0$. In all cases of $R = 0$ (Stokes flow) there is an exact solution in terms of Fourier series. Thus for small enough values of R, a series method of solution in the form of Fourier expansions

$$\psi(r,\theta) = \tfrac{1}{2}F_0(r) + \sum_{n=1}^{\infty}\{F_n(r)\cos n\theta + f_n(r)\sin n\theta\} \tag{6}$$

$$\zeta(r,\theta) = \tfrac{1}{2}G_0(r) + \sum_{n=1}^{\infty}\{G_n(r)\cos n\theta + g_n(r)\sin n\theta\} \tag{7}$$

can be constructed. This method was used by Dennis [16] in the case of the symmetrical problem of Fig. 1, in which only the terms $f_n(r)$ and $g_n(r)$ in (6) and (7) are not identically zero. Wang and Wu used a series of the form (7) for the vorticity, but used Fourier series for the individual velocity components (v_r, v_θ) in place of (6). The series were used to evaluate integrals arising in their integral-relation method of approach. The present approach uses the same method of reduction of (3) and (4) to ordinary differential equations in $f_n(r)$, $F_n(r)$, $g_n(r)$, $G_n(r)$ used by Dennis in the simpler case of Fig. 1, but employs integral conditions to calculate the boundary vorticity rather than the local finite-difference approximations used in Ref. [16], in some sense providing a bridging method between these methods and that of Wang and Wu.

The boundary conditions for the problem of Fig. 1 are that at $r = 1$,

$$\begin{aligned} \psi &= \theta/\alpha && \text{for } 0 \leqslant \theta \leqslant \alpha; \\ \psi &= 1 && \text{for } \alpha \leqslant \theta \leqslant \pi - \alpha; \\ \psi &= (\pi - \theta)/\alpha && \text{for } \pi - \alpha \leqslant \theta \leqslant \pi; \end{aligned} \tag{8a}$$

$$\partial\psi/\partial r = 0 \qquad \text{for } 0 \leqslant \theta \leqslant \pi. \tag{8b}$$

For the problem of Fig. 2 the corresponding conditions are that at $r = 1$

$$\begin{aligned} \psi &= (\theta - \alpha')/\varepsilon' && \text{for } \alpha' - \varepsilon' \leqslant \theta \leqslant \alpha' + \varepsilon'; \\ \psi &= 1 && \text{for } \alpha' + \varepsilon' \leqslant \theta \leqslant \beta - \varepsilon; \\ \psi &= (\beta - \theta)/\varepsilon && \text{for } \beta - \varepsilon \leqslant \theta \leqslant \beta + \varepsilon \\ \psi &= -1 && \text{for } \beta + \varepsilon \leqslant \theta \leqslant 2\pi + \alpha' - \varepsilon'; \end{aligned} \tag{9a}$$

$$\partial\psi/\partial r = 0 \qquad \text{for } 0 \leqslant \theta \leqslant 2\pi. \tag{9b}$$

The solution of the problem with the conditions of Eqs. (8) requires only the terms involving sines in Eqs. (6) and (7) while the solution subject to Eq. (9) requires all the terms, including the cosines. In both cases the boundary vorticity is calculated using integral conditions deduced from Eq. (3) with Eqs. (8) and (9). These conditions are given, together with the mathematical formulation, in the next section.

3. FORMULATION IN TERMS OF FOURIER SERIES

On substitution of the series (6) and (7) into Eqs. (3) and (4) we obtain, following the usual methods of Fourier analysis, sets of ordinary differential equations governing the Fourier coefficients. If we consider first Eq. (3) we obtain

$$F_0'' + r^{-1}F_0' = G_0, \tag{10}$$

where the prime denotes differentiation with respect to r. It follows that if $F_0'(0)$ is finite,

$$rF_0' = \eta(r), \tag{11}$$

where

$$\eta(r) = \int_0^r \xi G_0(\xi)\, d\xi \tag{12}$$

and then that

$$F_0(r) = F_0(1) - \int_r^1 \frac{\eta(\xi)}{\xi}\, d\xi. \tag{13}$$

For $n \neq 0$ we have the sets of equations

$$F_n'' + r^{-1}F_n' - r^{-2}n^2F_n = G_n, \tag{14a}$$

$$f_n'' + r^{-1}f_n' - r^{-2}n^2f_n = g_n, \tag{14b}$$

together with boundary conditions which will be stated later.

The equations governing the functions appearing in the series (7) for ζ are similarly obtained from (4). It is found that $G_0(r)$ satisfies

$$G_0'' + r^{-1}G_0' = r^{-1}\sigma'(r), \tag{15}$$

where

$$\sigma(r) = R \sum_{m=1}^{\infty} m(f_m G_m - F_m g_m). \tag{16}$$

If $G_0'(0)$ is finite, then

$$rG_0' = \sigma(r) \tag{17}$$

and, further,

$$G_0(r) = G_0(1) - \int_r^1 \frac{\sigma(\xi)}{\xi}\, d\xi. \tag{18}$$

For the functions $G_n(r)$ and $g_n(r)$ when $n \neq 0$ we have, respectively, the equations

$$G_n'' + r^{-1}G_n' - r^{-2}n^2G_n = \lambda_n(r), \tag{19a}$$

$$g_n'' + r^{-1}g_n' - r^{-2}n^2g_n = \mu_n(r), \tag{19b}$$

where the functions on the right-hand sides are defined respectively by

$$\begin{aligned}\lambda_n(r) = \frac{R}{2r}\Big\{ & nf_n G_0' + \sum_{m=1}^{\infty} mf_m[G_{m+n}' + G_{|m-n|}'] \\ & + \sum_{m=1}^{\infty} f_m'[(m+n)\, G_{m+n} + (m-n)\, G_{|m-n|}] \\ & - \sum_{m=1}^{\infty} mF_m[g_{m+n}' - \operatorname{sgn}(m-n)\, g_{|m-n|}'] - nF_0' g_n \\ & - \sum_{m=1}^{\infty} F_m'[(m+n)\, g_{m+n} + |m-n|\, g_{|m-n|}]\Big\},\end{aligned} \tag{20}$$

$$\begin{aligned}\mu_n(r) = \frac{R}{2r}\Big\{ & -nF_n G_0' - \sum_{m=1}^{\infty} mF_m[G_{|m-n|}' - G_{m+n}'] + nF_0' G_n \\ & + \sum_{m=1}^{\infty} F_m'[(m+n)\, G_{m+n} - (m-n)\, G_{|m-n|}] \\ & + \sum_{m=1}^{\infty} mf_m[g_{m+n}' - \operatorname{sgn}(m-n)\, g_{|m-n|}'] \\ & - \sum_{m=1}^{\infty} f_m'[|m-n|\, g_{|m-n|} - (m+n)\, g_{m+n}]\Big\}.\end{aligned} \tag{21}$$

In Eqs. (20) and (21), $\operatorname{sgn}(m-n)$ denotes the sign of $m-n$ with $\operatorname{sgn}(0)=0$.

Equations (14) and (19) form sets of second-order ordinary differential equations and each requires two boundary conditions. It is assumed that at $r=0$ the stream function and vorticity must be finite and unique (single-valued). If we multiply both sides of either (6) or (7) by $\cos n\theta$ or $\sin n\theta$ and integrate with respect to θ around an infinitesimally small circle centred at the origin, it follows that

$$f_n(0) = g_n(0) = 0, \qquad F_n(0) = G_n(0) = 0 \qquad (n = 1, 2, 3, \ldots). \tag{22}$$

Conditions for $F_0(0)$ and $G_0(0)$ cannot be found in this way but when $n=0$ we need only know $F_0(1)$ in (13) and $G_0(1)$ in (18) to determine $F_0(r)$ and $G_0(r)$, assuming that the

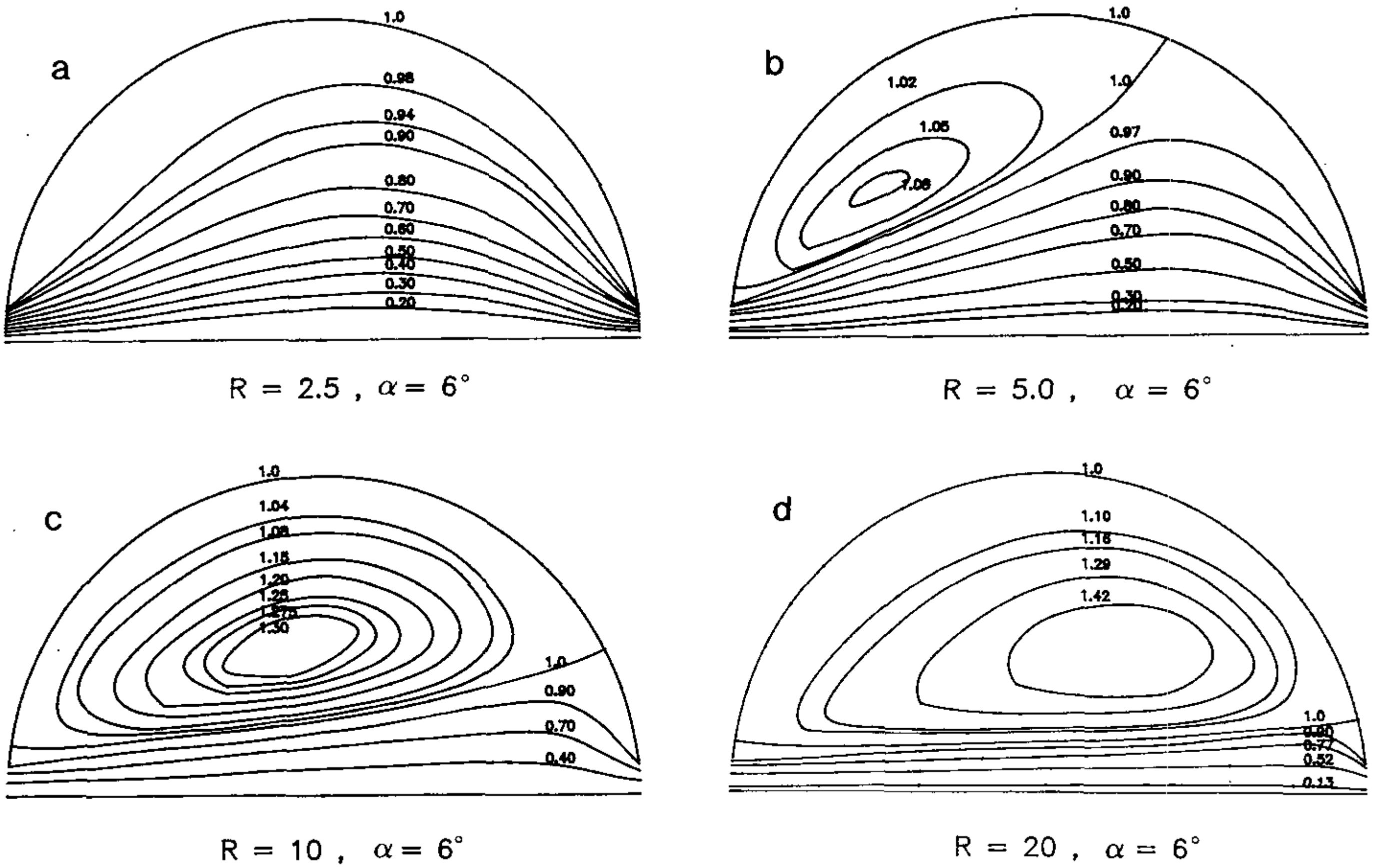

FIG. 3. Streamlines for the symmetrical inflow–outflow problem for $\alpha = 6°$ with $h = \frac{1}{40}$, $N = 80$: (a) $R = 2.5$; (b) $R = 5$; (c) $R = 10$; (d) $R = 20$.

5. RESULTS

The main object of this study is to show that satisfactory results can be obtained in the problem considered by using the series truncation formulation in conjunction with integral conditions of global type. These conditions are used to calculate the boundary vorticity, thereby avoiding the use of the usual local approximations when finite-difference methods are used. To this end we have computed again some of the results presented by Dennis [16], namely the symmetrical flow of Fig. 1 for the case $\alpha = 6°$, $R = 2.5$, 5, 10, 20. The streamlines for the most accurate results computed are shown in Figs. 3a–d in these four cases, respectively. The most accurate results used $h = \frac{1}{40}$, $N = 80$, but solutions were also obtained for other grid sizes and values of N (the maximum number of terms). Actually, $N = 80$ is far in excess of the number of terms needed to obtain the graphical accuracy displayed in Fig. 3. As far as grid sizes are concerned, a comparison of the results for $R = 20$ for the two grid sizes $h = \frac{1}{20}$, $\frac{1}{40}$ may be made by means of Fig. 4 which shows streamlines for $R = 20$, $h = \frac{1}{20}$, $N = 80$.

For detailed comparison purposes we have displayed in Fig. 5 the results obtained by Dennis [16] for $R = 0, 2, 5, 10,$ using $h = \frac{1}{20}$, $N = 40$. Figures 5c, d are seen to be in extremely good comparison with Figs. 3b, c, respectively. It is more difficult to compare with the results given by Wang and Wu because they have not identified precisely their Reynolds number Re. However, they have based the non-dimensionalization of the inlet normal velocity on a dimensionless radial component,

$$v_r = -1(\pi - \alpha < \theta < \pi + \alpha), \qquad v_r = 1(-\alpha < \theta < \alpha),$$

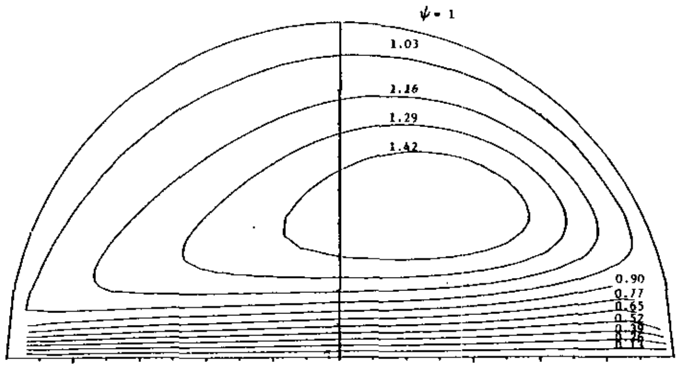

FIG. 4. Streamlines for the symmetrical inflow–outflow problem for $\alpha = 6°$, $h = \frac{1}{20}$, $N = 80$, $R = 20$.

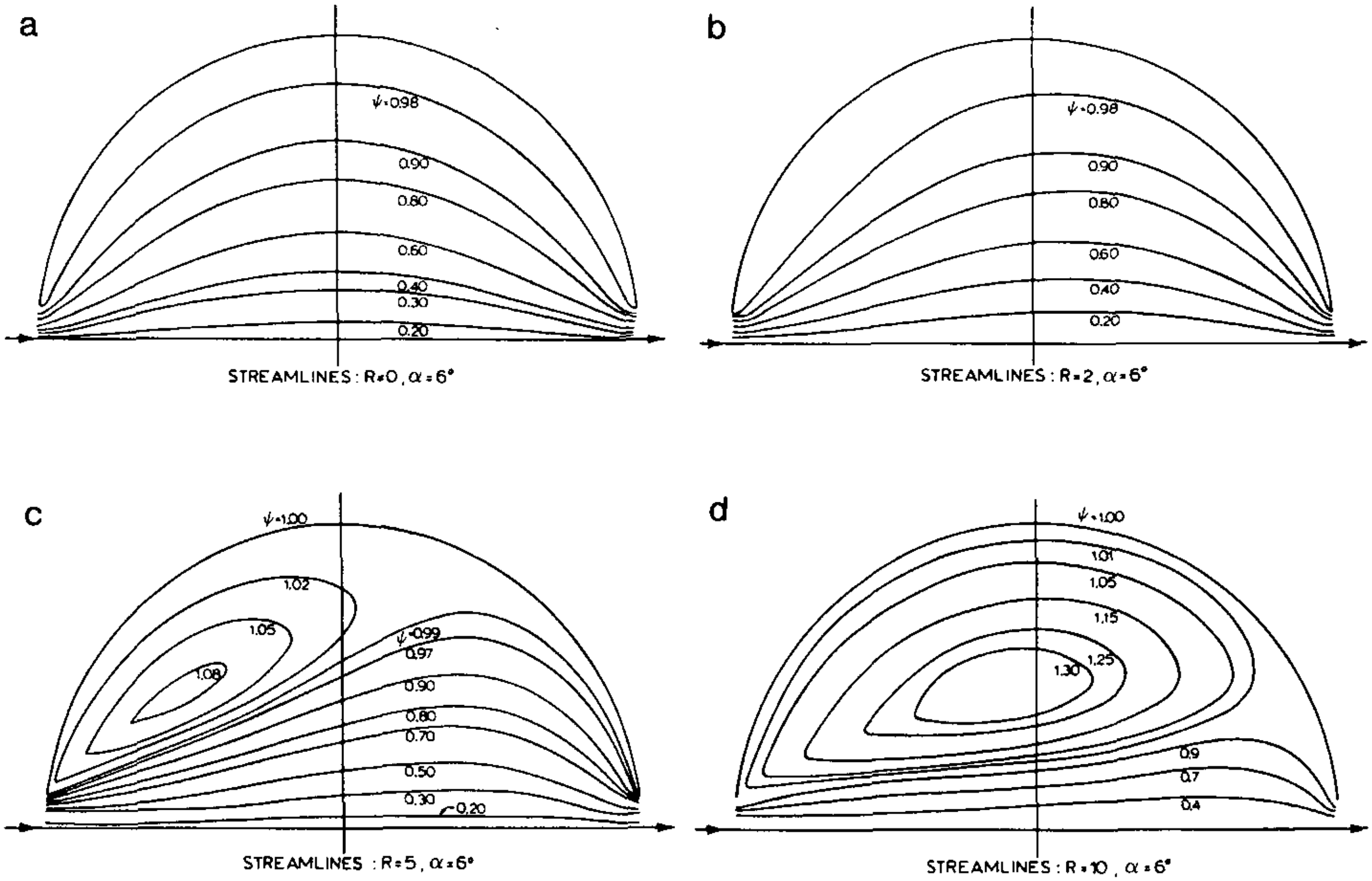

FIG. 5. Streamlines for the symmetrical inflow–outflow problem computed by Dennis [12] for $\alpha = 6°$, $h = \frac{1}{20}$, $N = 40$: (a) $R = 0$; (b) $R = 2$; (c) $R = 5$; (d) $R = 10$.

so their Reynolds number is probably $\mathrm{Re} = Ua/\nu$ compared with our Reynolds number of Eq. (5). We have used the same Reynolds number as Dennis [16] (note that the inlet–outlet velocity of Dennis [16], Fig. 1 should be shown as $\alpha v_r = \pm 1$ rather than $v_r = \pm 1$). With this understanding, $\mathrm{Re} = R/\varepsilon = 30R/\pi$ for the present example, so the Reynolds numbers of Wang and Wu (who also consider $\alpha = 6°$) should be approximately 10 times ours. This seems to be borne out in Fig. 6, where the streamlines for Wang and Wu's solution at Re = 0, 20, 50, 100 are given (taken from the paper cited). All four parts agree extremely well with the corresponding parts of Fig. 5 and parts c and d are in good agreement with parts b and c, respectively, of Fig. 3. There is, therefore, extremely good mutual comparison between all three sets of results. This indicates that all methods of approach are viable. The computer times for calculating the present solutions are quite modest. In fact it is probably quite comparable to solve the sets of Eqs. (10), (14), (15), and (19) by numerical methods in the manner indicated with the solution procedure of Wang and Wu, who evaluate numerically the integrals which define the velocity components and vorticity in their approach. Both methods employ iterative techniques and, for the higher values of R, both methods employ some form of relaxation procedure of the type (29) in the iterative procedure.

In the basic problem considered by Rayleigh (for $R = 0$ only) the value $\alpha = 0$ was taken, which corresponds to a limiting case in which fluid is injected and removed over a vanishingly small arc with a correspondingly large velocity. There are no other published results for this case of $\alpha \to 0$ and it is interesting to see what the effect of this reduction of α has on the results. In Fig. 7 we show some streamlines for $R = 2.5$, 5, 10 in the Rayleigh case $\alpha = 0$, obtained using $h = \frac{1}{40}$, $N = 40$. Compared with the $\alpha = 6°$ case the flow tends to be more displaced in the downstream region for the comparable Reynolds number and the recirculating region, when it appears, is generally more rapid. Both of these effects may be expected with the more confined injection of the fluid.

Finally, we have considered the Reynolds numbers

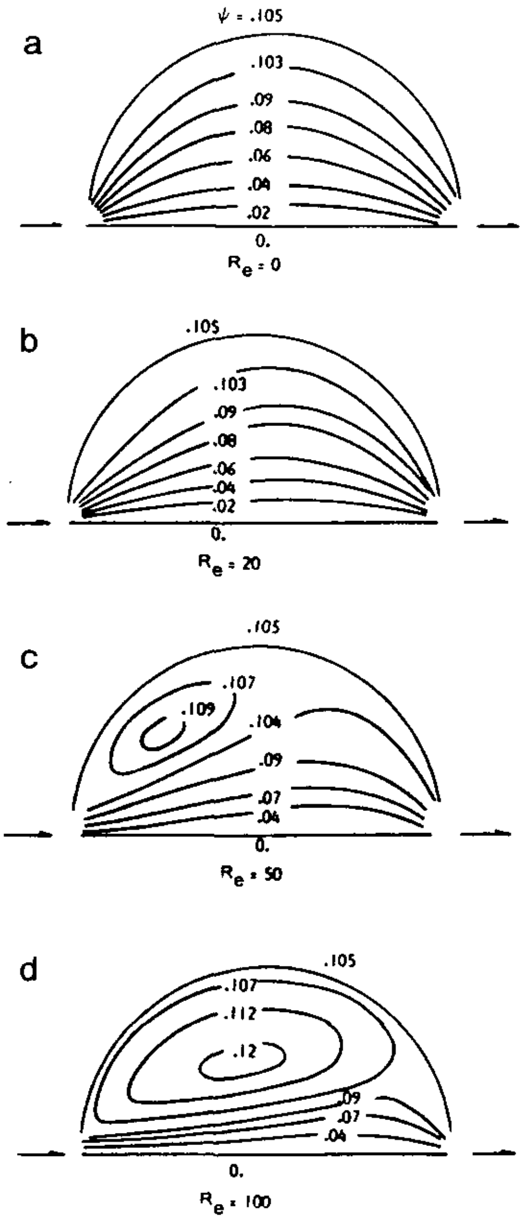

FIG. 6. Streamlines for the symmetrical inflow–outflow problem computed by Wang and Wu (reproduced, by permission of the publisher, from Ref. [1, copyright © AIAA 1985]) for $\alpha = 6°$: (a) Re = 0; (b) Re = 20; (c) Re = 50; (d) Re = 100.

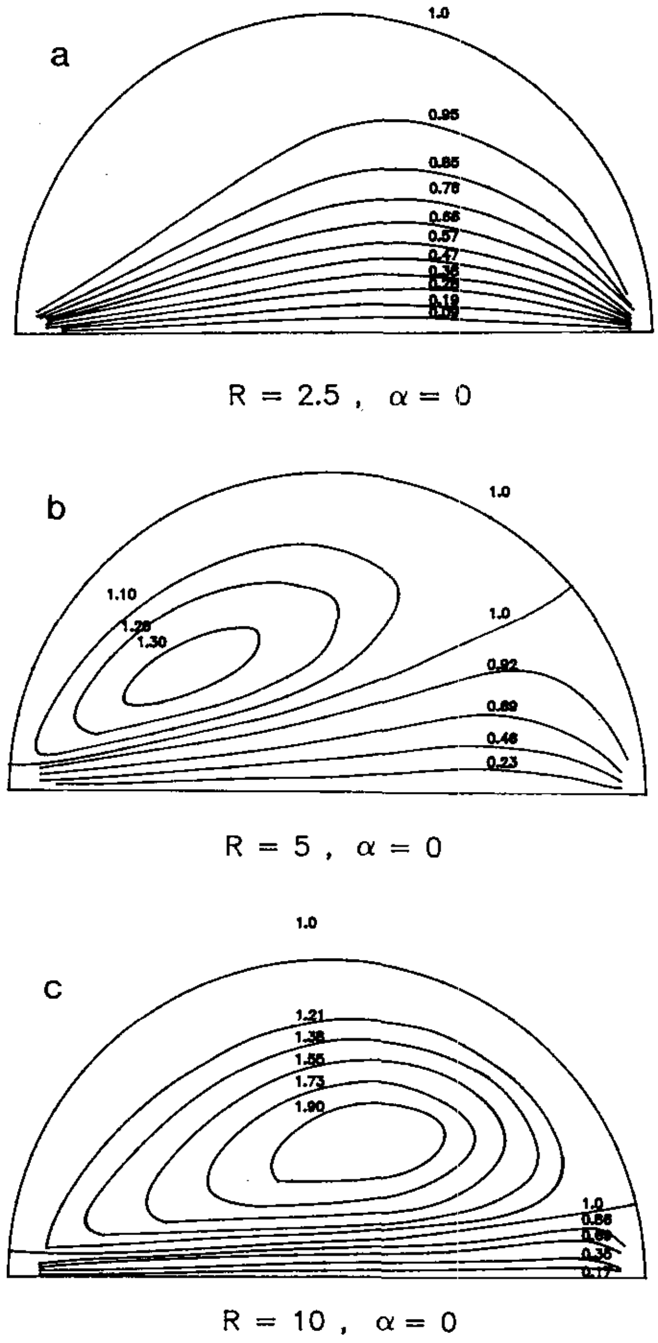

FIG. 7. Streamlines for the symmetrical inflow–outflow problem for $\alpha = 0$, $h = \frac{1}{40}$, $N = 40$: (a) $R = 2.5$; (b) $R = 5$; (c) $R = 10$.

$R = 2.5$ and 5 in one case in which the flow is asymmetrical, simply to test the method and compare the results with previous computations. The parameters for these cases (Fig. 2) are

$$N = 80, \qquad h = \frac{1}{20}, \qquad \alpha' = \frac{\pi}{8},$$

$$\beta = \pi, \qquad \varepsilon = \varepsilon' = \frac{\pi}{32}.$$

These are cases considered by Mills [13] to illustrate his integral relation method. Our computed streamlines are shown in Fig. 8. The corresponding streamlines obtained by Mills at his $R_l = 2.5$ (which seems to be the same as our R in Eq. (5)) are shown in Fig. 9a. They show only one region of separation rather than the two found in our solution. However, Fig. 9b shows Mills results for $R = 5$ and our results seem to be much nearer to this diagram and certainly of the same character. It is difficult to say which are the most

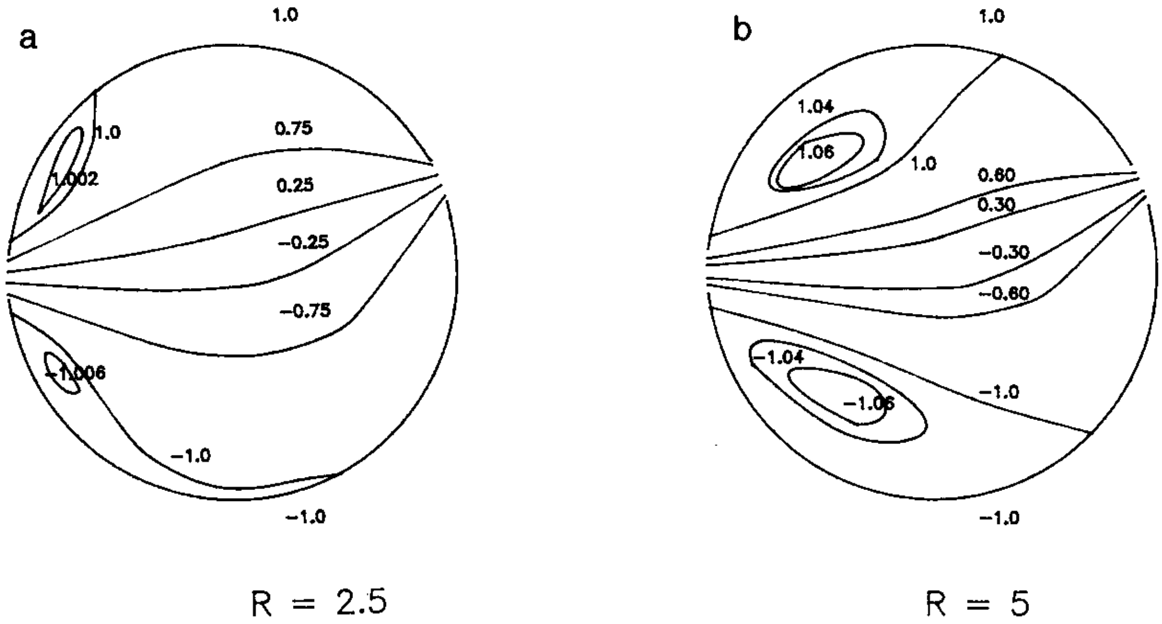

FIG. 8. Streamlines for the asymmetrical inflow–outflow problem with $\alpha' = \pi/8$, $\beta = \pi$, $\varepsilon = \varepsilon' = \pi/32$: (a) $R = 2.5$; (b) $R = 5$.

accurate, although it must be pointed out that the results of Mills were computed with a radial grid of only $h = \frac{1}{10}$.

In conclusion, we have presented here a method which lies in between the purely finite-difference method, with calculation of boundary vorticity locally, and the integral relation method of Wang and Wu. The method uses the same type of integral conditions as those employed by Wang and Wu, but the general method of computation uses numerical solution of the governing differential equations rather than evaluation of solutions in integral form. The results obtained are quite comparable in the trial cases considered. It seems, therefore, that both methods are worth consideration as alternatives to a method in which local calculation of the boundary vorticity is replaced by conditions of global type. The method used by Mills [13] is again different and it seems also to be worthy of further consideration. Further details of the present investigation are to be found in the Ph.D. thesis of M. Ng [19].

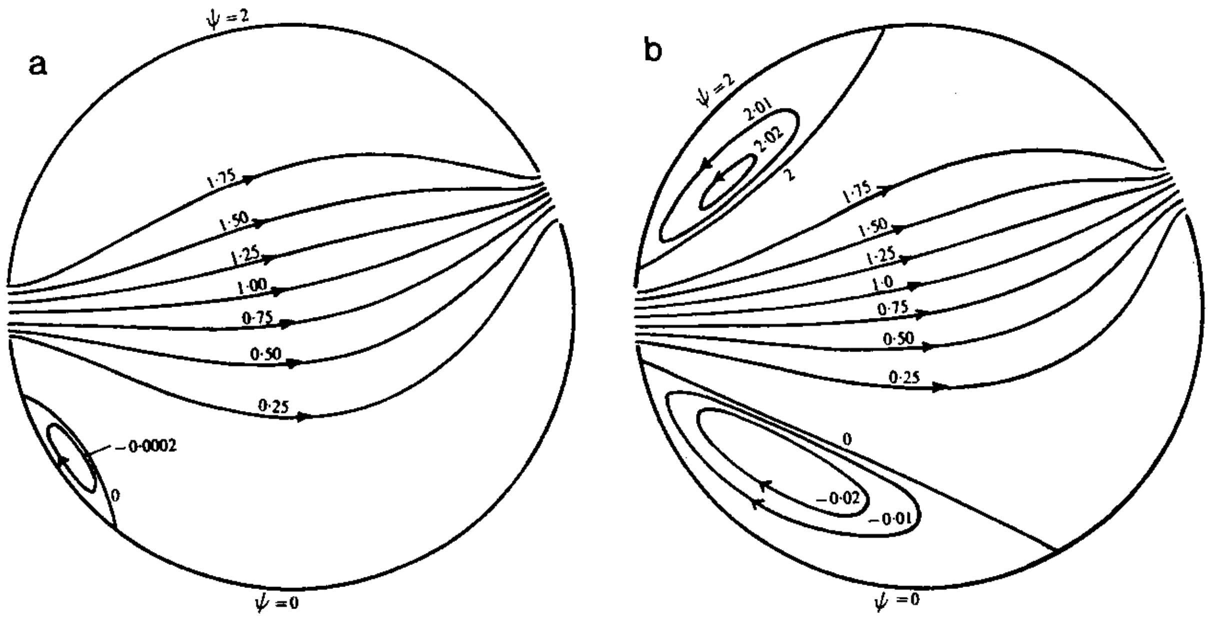

FIG. 9. Streamlines for the asymmetrical inflow–outflow problem computed by Mills [13] with $\alpha' = \pi/8$, $\beta = \pi$, $\varepsilon = \varepsilon' = \pi/32$: (a) $R_l = 2.5$; (b) $R_l = 5$.

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